

A Semiclassical Heat Kernel Proof of the Poincaré-Hopf Theorem

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Abstract

We treat the Witten operator on the de Rham complex with semiclassical heat kernel methods to derive the Poincaré-Hopf theorem and degenerate generalizations of it. Thereby, we see how the semiclassical asymptotics of the Witten heat kernel are related to approaches using the Thom form of Mathai and Quillen.

1. Introduction

Wenn man 2 Wege hat, so muss man nicht bloss diese Wege gehen oder neue suchen, sondern dann das ganze zwischen den beiden Wegen liegende Gebiet erforschen.

*Given two routes, it is not right to take either of these two or to look for a third; it is necessary to investigate the area lying between the two routes.*¹

(David Hilbert)

The connection between semiclassical Analysis and Morse theory was discovered by Witten [Wit82] about thirty years ago and got a lot of attention ever since. The physical idea is the following: The Laplace part of a Schrödinger operator $\hbar^2\Delta + V$ is responsible for diffusion, while the potential V causes concentration at its cavities. Therefore, when simultaneously taking limits $\hbar \rightarrow 0$ and $t \rightarrow \infty$ in some appropriate way, one expects the particles to concentrate near the minima of the potential. With Morse theory on the other

¹Cited in [Hug06]

hand, one can obtain topological information of the underlying space by investigating the critical points of a given function.

Witten intertwines the Euler operator acting on differential forms with a vector field X , this perturbation depending on a small parameter \hbar . That way he obtains a Schrödinger type operator whose semiclassical eigenfunction approximations he uses to construct the Morse complex.

The Morse inequalities directly imply the Poincaré-Hopf theorem, which states that the Euler characteristic of M is determined by the critical points of X , more precisely

$$\chi(M) = \sum_{p \in C_X} (-1)^{\nu(p)},$$

where C_X is the set of critical points of X and the index $\nu(p)$ of a critical point is equal to the number of negative eigenvalues of ∇X .

Another way to get this theorem uses the Thom form U of Mathai and Quillen: By the transgression formula for the Thom form, the pullbacks of U along any two vector fields are cohomologous; on the other hand, the Euler class of M is the pullback of U along the zero vector field. The Pullback of U along the vector field $X_t := t^{1/2}X$ gives a differential form on M and the Poincaré-Hopf theorem follows then by evaluating X_t^*U with the method of stationary phase [BGV96, Thm. 1.56].

These proofs seem conceptually very different at first: Witten uses the low-lying eigenfunctions of his operator to construct a complex homotopic to the de Rham complex whence the theorem follows by an argument of homological algebra, while the other proof derives an integral formula that interpolates between the Gauss-Bonnet-Chern theorem and Poincaré-Hopf. In this article, we show that in fact one can use the semiclassical asymptotics [BP10] of the heat kernel of Witten's operator to recover the interpolation formula appearing in the Thom form proof.

More precisely, from the semiclassical heat kernel asymptotics, we derive the integral formula

$$\chi(M) = \int_M \alpha(t) e^{-t|X|^2} \quad \text{for all } t > 0,$$

where $\alpha(t)$ is some function depending polynomially on t . We then use ideas from Getzler's proof of the local index theorem [Get85] to explicitly calculate $\alpha(t)$ in terms of the curvature of M and Taylor coefficients of X . It turns out that the integrand above is nothing but the Thom form, pulled back via X_t .

We show that in the limit $t \downarrow 0$, the integral formula above yields the Gauss-Bonnet-Chern theorem and in the limit $t \uparrow \infty$, the integral can be evaluated with the method of stationary phase to obtain the Poincaré-Hopf theorem. In this sense, the t -dependent integral formula above interpolates between two classical theorems.

More general, by allowing the critical set to be a union of submanifolds of M , we obtain a degenerate version of the Poincaré-Hopf theorem (Thm 6.2). Let us remark that the degenerate Morse inequalities (Morse-Bott inequalities) that imply the degenerate Poincaré-Hopf theorem have been proved with heat kernel methods by Bismut [Bis86],

but the integrands are not explicitly calculated in terms of curvature. For other treatments of the Morse-Bott inequalities, see [Bot54] or [AB95]), for example.

This article is organized as follows: At first, we briefly review some basic notions of the Clifford algebra and the exterior algebra of a Euclidean vector space V . In section 3, we state the needed results about the semiclassical expansion of the heat kernel of Witten's operator. Afterwards, we introduce Getzler symbols and use them to calculate the relevant integrands. In section 5, we put these results together to prove the stated integral formula for the index and the consequences of it. The last section is dedicated to the degenerate case, in which the vector field X may have critical submanifolds instead of just critical points. This needs a longer calculation because of the more complicated nature of the corresponding stationary phase expansion.

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2. The Clifford Symbol on the Exterior Algebra

We begin with a short recap of the filtrations, gradings and symbol maps associated to the exterior algebra.

In this section, let V be a Euclidean vector space of dimension n . The exterior algebra ΛV has two different gradings, we are interested in, a \mathbb{Z} -grading induced by the degree of forms and a \mathbb{Z}_2 -grading, where the even (odd) part is the space of even-degree (odd-degree) forms. The latter gives ΛV the structure of a superalgebra; the associated grading operator Ξ is by definition the operator that is the identity on even-degree forms and minus the identity on odd-degree forms.

For $v \in V$, we define the Clifford multiplications $\Lambda V \rightarrow \Lambda V$ by

$$\mathbf{c}(v) = \varepsilon(v) - \iota(v), \quad \mathbf{b}(v) = \varepsilon(v) + \iota(v).$$

Here, ε and ι denote exterior and interior multiplication, where to define the latter, we use the euclidean structure on V . For $v, w \in V$, we have the relations

$$[\mathbf{c}(v), \mathbf{c}(w)]_s = -\langle v, w \rangle, \quad [\mathbf{b}(v), \mathbf{b}(w)]_s = \langle v, w \rangle, \quad [\mathbf{c}(v), \mathbf{b}(w)]_s = 0,$$

where $[\mathbf{c}(v), \mathbf{c}(w)]_s = \mathbf{c}(v)\mathbf{c}(w) + \mathbf{c}(w)\mathbf{c}(v)$ denotes the super commutator. From this follows that the endomorphism space $\text{End}(\Lambda V)$ is generated as an algebra by the elements $\mathbf{c}(v), \mathbf{b}(v), v \in V$. It is easy to check that the grading operator can be represented as

$$\Xi = (-1)^{\lfloor n/2 \rfloor} \mathbf{c}^1 \dots \mathbf{c}^n \mathbf{b}^1 \dots \mathbf{b}^n,$$

where we wrote $\mathbf{c}^j = \mathbf{c}(e^j)$, $\mathbf{b}^j = \mathbf{b}(e^j)$ for an orthonormal basis e^1, \dots, e^n of V . Ξ does not depend on the choice of this orthonormal basis.

The algebra $\text{End}(\Lambda V)$ has a filtration and a bi-filtration, which we will both use: An element $A \in \text{End}(\Lambda V)$ has (Clifford-) bi-order (k, l) or lower if it can be written as

$$A = \sum_{|I| \leq k} \sum_{|J| \leq l} A_{IJ} \mathbf{c}^I \mathbf{b}^J, \quad (2.1)$$

where we wrote $\mathbf{c}^I := \mathbf{c}^{i_1} \cdots \mathbf{c}^{i_m}$ for a multiindex $I = (i_1, \dots, i_m)$. We say that A has (Clifford-) order k or lower if it has bi-order (k, n) or lower.

For such an element A of order k , we define its k -th Clifford symbol by

$$\phi_k(A) := \phi_{k, \bullet}(A) := \sum_{|I|=k} \sum_{|J| \leq n} A_{IJ} e^I \hat{\otimes} \mathbf{b}^J \in \Lambda V \hat{\otimes} \text{End}(\Lambda V),$$

where $\hat{\otimes}$ denotes the super tensor product. The Clifford bi-symbol of an element A of bi-order (k, l) will be defined as

$$\phi_{k,l}(A) := \sum_{|I|=k} \sum_{|J|=l} A_{IJ} e^I \hat{\otimes} e^J \in \Lambda V \hat{\otimes} \Lambda V.$$

It is straightforward to check that all these definitions do not depend on the choice of orthonormal basis.

Definition 2.1. The supertrace on $\text{End}(\Lambda V)$ is defined by $\text{str}(A) := \text{tr}(\Xi A)$, where Ξ is the grading operator.

Proposition 2.2. If A has order (k, l) with $k < n$ or $l < n$, then $\text{str}(A) = 0$. On the other hand, $\text{str}(\Xi) = 2^n$.

Proof. Let $A = \mathbf{c}^{j_1} \cdots \mathbf{c}^{j_k} \mathbf{b}^{i_1} \cdots \mathbf{b}^{i_l} \in \text{End}(\Lambda V)$ with $k < n$ and let $u \notin \{j_1, \dots, j_k\}$. Then straightforward calculation shows

$$[\mathbf{c}^u \mathbf{c}^{j_1} \cdots \mathbf{c}^{j_k} \mathbf{b}^{i_1} \cdots \mathbf{b}^{i_l}, \mathbf{c}^u]_s = (-1)^{k+l} 2 \mathbf{c}^{j_1} \cdots \mathbf{c}^{j_k} \mathbf{b}^{i_1} \cdots \mathbf{b}^{i_l},$$

so that A is a super-commutator. However, str vanishes on super-commutators. In the case that $k = n$ and $l < n$, replace \mathbf{c}^u by \mathbf{b}^u for some $u \notin \{i_1, \dots, i_l\}$.

Finally, by definition, $\text{str}(\Xi) = \text{tr}(\Xi^2) = \text{tr}(\text{id}) = 2^n$. \square

Corollary 2.3. We have

$$\text{str}(A) = (-1)^{\lfloor n/2 \rfloor} 2^n \langle \phi_{n,n}(A), \text{vol} \hat{\otimes} \text{vol} \rangle$$

where $\text{vol} = e^1 \wedge \cdots \wedge e^n$ for an (oriented) orthonormal basis e^1, \dots, e^n and $\langle \cdot, \cdot \rangle$ is chosen such that $\text{vol} \hat{\otimes} \text{vol}$ has norm 1.

Remark 2.4. Notice that this does not depend on an orientation of M . Changing the orientation turns vol into $-\text{vol}$, but the two signs cancel each other.

Proof. The symbol $\phi_{n,n}$ picks out the (n, n) -form part and the pairing with $\text{vol} \hat{\otimes} \text{vol}$ extracts the coefficient. However, the supertrace is 2^n times this coefficient, as $\text{str}(\Xi) = (-1)^{\lfloor n/2 \rfloor} 2^n$. \square

3. The Semiclassical Heat Kernel Expansion

Definition 3.1. Let M be a Riemannian manifold and \mathcal{E} a real or complex vector bundle equipped over M with a scalar product (or Hermitian form respectively). We say that an operator of the form

$$H_{\hbar} = \hbar^2 L + \hbar W + V, \quad (3.1)$$

is of Schrödinger type, if L is a formally self-adjoint operator of Laplace type, i.e. locally, it has the form

$$L = -\text{id}_{\mathcal{E}} \sum_{ij=1}^n g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \text{lower order terms},$$

and V and W are symmetric endomorphism fields (we adopt the sign convention such that the eigenvalues of Laplace type operators tend to $+\infty$).

Example 3.2 (The Witten operator). Let $\mathcal{E} = \Lambda T^*M$. For a one-form $\xi \in \Omega^1(M)$, define

$$D_{\hbar} = \hbar D + \mathbf{b}(\xi), \quad \text{where} \quad D = d + d^*.$$

Then D_{\hbar}^2 is of Schrödinger type, as straightforward calculation shows that in local coordinates, we have

$$W = [D, \mathbf{b}(\xi)] = \sum_i \mathbf{c}(\text{d}x^i) \mathbf{b}(\nabla_i \xi), \quad \text{and} \quad V = |\xi|^2. \quad (3.2)$$

Witten [Wit82] uses this operator, but with a different normalization ($\hbar \mapsto t^{-1}$). Also see [Roe98, p. 125].

Suppose that M is compact. Then it is well-known that for each $\hbar > 0$, a Schrödinger type operator H_{\hbar} is an unbounded operator in $L^2(M, \mathcal{E})$ which is self-adjoint on the Sobolev space $H^2(M, \mathcal{E})$ and has eigenvalues tending to $+\infty$. For $t > 0$, the operator $e^{-tH_{\hbar}}$ (defined by functional calculus) has a smooth integral kernel

$$k_{\hbar} \in \Gamma^{\infty}(M \times M \times (0, \infty), \mathcal{E}^* \boxtimes \mathcal{E}). \quad (3.3)$$

that depends smoothly on \hbar [BGV96, Thm. 2.48]. Here, $\mathcal{E}^* \boxtimes \mathcal{E}$ is the fiberwise product, i.e. $(\mathcal{E}^* \boxtimes \mathcal{E})_{(p,q)} = \mathcal{E}_p^* \otimes \mathcal{E}_q \cong \text{Hom}(\mathcal{E}_p, \mathcal{E}_q)$.

On any complete Riemannian manifold M , we call

$$e_{\hbar}(p, q, t) = (4\pi t \hbar^2)^{-n/2} \exp\left(-\frac{1}{4t \hbar^2} d(p, q)^2\right)$$

the Euclidean heat kernel, because $k_{\hbar} = e_{\hbar}$ if L is the usual Laplace operator on functions in euclidean space and $W = V = 0$. Here, $d(p, q)$ is the Riemannian distance between p and q so that e_{\hbar} is smooth on the set

$$M \bowtie M = \{(p, q) \in M \times M \mid p \text{ is not a cut-point of } q\}.$$

The \hbar -asymptotics of the heat kernel k_\hbar are developed in [BP10]. We briefly recall the relevant ideas: Because k_\hbar is the solution kernel to the heat equation, we have $(\partial_t + H_\hbar)k_\hbar = 0$ for each $q \in M$ and $\hbar > 0$, where H_\hbar is supposed to act on the first entry of k_\hbar . If on $M \bowtie M$ we make the ansatz

$$k_\hbar \sim e_\hbar \sum_{j=0}^{\infty} \hbar^j \Phi_j \quad (3.4)$$

for an asymptotic expansion of k_\hbar in \hbar , we can formally (i.e. termwise) apply the heat operator $(\partial_t + H_\hbar)$ to this expression and straightforward calculation shows that the result is again of the form e_\hbar times some power series in \hbar . Ordering the result by powers of \hbar shows that for each $q \in M$, the Φ_j have to fulfill the recursive transport equations

$$\left(t \frac{\partial}{\partial t} + \nabla_{\mathcal{V}} + G + tV \right) \Phi_j(\cdot, q) = -tW\Phi_{j-1}(\cdot, q) - tL\Phi_{j-2}(\cdot, q), \quad (3.5)$$

where

$$\mathcal{V} = \frac{1}{2} \text{grad}[d(\cdot, q)^2], \quad G = - \left(\frac{n}{2} + \frac{1}{4} \Delta[d(\cdot, q)^2] \right). \quad (3.6)$$

Therefore one makes the following definition.

Definition 3.3. Let $\sum_{j=0}^{\infty} \hbar^j \Phi_j$ be a formal power series with coefficients Φ_j in the space $\Gamma^\infty(M \bowtie M \times [0, \infty), \mathcal{E}^* \boxtimes \mathcal{E})$. Then the formal expression

$$\hat{k}_\hbar := e_\hbar \sum_{j=0}^{\infty} \hbar^j \Phi_j$$

is called semiclassical heat kernel expansion if for all $q \in M$, the coefficients Φ_j fulfill the recursive transport equations (3.5) with the initial condition $\Phi_0(q, q, 0) = \text{id}_{\mathcal{E}_q}$.

Regarding this, we have the following theorem.

Theorem 3.4 (Bär, Pfäffle). [BP10, Lemma 3.1] *Let M be a complete Riemannian manifold, \mathcal{E} a vector bundle with scalar product over M and H_\hbar be of Schrödinger type. Then there exists a semiclassical heat kernel expansion.*

Furthermore, for each $q \in M$, the first transport equation (3.5) has a unique solution for each prescribed value for $\Phi_0(q, q, 0)$, while for $j > 0$, the j -th transport equation has a unique solution for each right side.

The relation between the formal heat kernel and the true heat kernel is the following.

Theorem 3.5 (Bär, Pfäffle). [BP10, Thm. 3.3] *Let M be compact. Then for all $T > 0$, $m, k \in \mathbb{N}_0$ and $N > n + m + k$, there exist constants $C > 0$, $\hbar_0 > 0$ such that*

$$\sup_{t \in (0, T]} \left\| \frac{\partial^k}{\partial t^k} \left(k_\hbar - \chi e_\hbar \sum_{j=0}^N \hbar^j \Phi_j \right) \right\|_{C^m(M \times M)} \leq C \hbar^{2N - 2n - 2m - 2k + 1}$$

for all $\hbar < \hbar_0$. Here, χ is a smooth cutoff function that is compactly supported in $M \bowtie M$ and $\chi \equiv 1$ on a neighborhood of the diagonal in $M \times M$.

Remark 3.6. In fact, Bär and Pfäffle [BP10] show 3.4 only for $W = 0$. In this case, $\Phi_j = 0$ whenever j is odd. While this is not true for non-zero W , the transport equations (3.5) can be solved just as well and the proof of Thm. 3.5 carries over basically without changes.

4. Getzler Symbols

This section is dedicated to the proof of the following theorem.

Theorem 4.1. *Let H_h be the Witten operator of example 3.2 acting on sections of ΛT^*M and let $e_h \sum_{j=0}^{\infty} h^j \Phi_j$ be the corresponding formal heat kernel. Then for each j , the coefficient Φ_j has Clifford bi-order at most (j, j) and*

$$\sum_{j=0}^n h^j \phi_{j,j}(\Phi_j) = \exp(-t \sigma_{0,0}(V) - t h \phi_{1,1}(W) t h^2 - \phi_{2,2}(\mathbf{F})). \quad (4.1)$$

Here,

$$\mathbf{F} = -\frac{1}{8} \sum_{ijkl} R_{ijkl} \mathbf{c}^i \mathbf{c}^j \mathbf{b}^k \mathbf{b}^l. \quad (4.2)$$

To establish a proof, we follow the ideas of Getzler's proof of the local Atiyah-Singer index theorem [Get85]: We define a symbol calculus on the space $\mathfrak{D}(M, \Lambda T^*M)$ of differential operators acting on sections of ΛT^*M that takes into account the Clifford order. Also see [Roe98] for further discussions of this.

Let M be a Riemannian manifold of dimension n . The algebra $\mathfrak{D}(M, \Lambda T^*M)$ of differential operators acting on sections of ΛT^*M has a natural filtration by order, and comes with the symbol map that assigns each operator its principal symbol. The bundle $\text{End}(\Lambda T^*M)$ already has a pointwise filtration and is equipped with the Clifford symbol

$$\phi_{\bullet} : \text{End}(\Lambda T^*M) \longrightarrow \mathcal{A} := \Lambda T^*M \hat{\otimes} \text{End}(\Lambda T^*M).$$

Now let us use all this data to define a new filtration on $\mathfrak{D}(M, \Lambda T^*M)$.

For $q \in M$, choose a chart x near q with $x(q) = 0$. If we additionally trivialize the bundle ΛT^*M by identifying fibers $\Lambda^k T_p^*M$ with $\Lambda^k T_q^*M$ for p near q , any differential operator $P \in \mathfrak{D}(M, \Lambda T^*M)$ has a Taylor series with respect to these choices,

$$P \sim \sum_{\alpha\beta} p_{\alpha\beta} x^\alpha \frac{\partial^{|\beta|}}{\partial x^\beta}, \quad \text{with } p_{\alpha\beta} \in \text{End}(\Lambda T_q^*M). \quad (4.3)$$

Of course, this Taylor series depends heavily on the two choices made. Its order and the principal term, however, do not, allowing us to define the following.

Definition 4.2 (Getzler Symbols). We say that P is of q -order k or less, if for each α and β , $p_{\alpha\beta}$ has order less or equal to $k + |\alpha| - |\beta|$ in the Clifford filtration of $\text{End}(\Lambda T_q^*M)$. In this case, its k -th q -symbol is

$$\sigma_k(P) = \sum_{j+|\beta|-|\alpha|=k} \phi_j(p_{\alpha\beta}) X^\alpha \frac{\partial^{|\beta|}}{\partial X^\beta}. \quad (4.4)$$

Here, X_j are the (Euclidean) coordinate functions on T_qM induced by the chart x . $\sigma_k(P)$ is a differential operator on T_qM with coefficients in the algebra $\mathcal{A}[T_qM]$, the space of \mathcal{A} -valued polynomials on T_qM ; we denote this space by $\mathfrak{P}(T_qM, \mathcal{A})$.

Remark 4.3. Taking the view that a symbol map is a homomorphism-like mapping from a filtered algebra into a graded algebra, this is indeed a symbol map: We equip $\mathcal{A}[T_qM]$ with the grading that assigns to $X^\alpha \partial^{|\beta|} / \partial X^\beta$ the degree $|\beta| - |\alpha|$ and to an element in $\Lambda^k T_q^*M \hat{\otimes} \text{End}(\Lambda T_q^*M)$ the degree k .

Remark 4.4. Let us make the essential observation that this symbol is in some sense a refinement of the Clifford filtration: If P is an endomorphism (i.e. a differential operator of order zero in the usual filtration) and it has q -order k or less, then its Clifford order is k or less as well, and $\phi_k(P)$ is given by evaluating $\sigma_k(P)$ at $X = 0$.

Proposition 4.5. *The definition above is independent of the choices made. Furthermore, we have*

$$\sigma_{j+k}(P \circ Q) = \sigma_j(P) \circ \sigma_k(Q) \quad (4.5)$$

whenever P is of q -order $\leq j$ and Q is of order $\leq k$.

Proof. If we have the well-definedness of the symbol map, (4.5) follows directly from the fact that the composition of differential operators in $\mathfrak{P}(T_qM, \mathcal{A})$ is defined the way it is. So let us check well-definedness. Let us temporarily denote by σ_k^x the k -th symbol with respect to a chart x . If we change from a chart x to a chart y , then the charts have Taylor expansions with respect to each other, namely

$$\begin{aligned} x &\sim Ay + \text{lower order terms} & \text{and} \\ y &\sim Bx + \text{lower order terms} \end{aligned}$$

for some $A \in \text{GL}(n)$ and $B = A^{-1}$. Therefore

$$\sigma_{|\alpha|}^x(x^\alpha) = X^\alpha = (AY)^\alpha = \sigma_{|\alpha|}^y((Ay)^\alpha) = \sigma_{|\alpha|}^y(x^\alpha),$$

because $(Ay)^\alpha - x^\alpha$ is of order lower than $|\alpha|$. A similar computation can be done for the differential operator $\partial^{|\beta|} / \partial x^\beta$, with the matrix A replaced by B .

Now under a trivialization, we understood a smooth identification of near-by fibers $\Lambda_p^k T^*M$ with $\Lambda^k T_q^*M$. A transformation T from one such identification to another is of course the identity at q , so the Taylor series of T with respect to any chart is $T \sim \text{id} + \text{lower order}$. Hence the symbol is independent of the choice of trivialization. \square

Getzler's main observation was that the q -symbol of a Dirac operator is a harmonic oscillator on $T_q M$, meaning the following.

Proposition 4.6. [Roe98, Prop. 12.17] *Let $D = d + d^*$. For every $q \in M$, D^2 has q -order 2 and its q -symbol with respect to orthogonal coordinates on $T_q M$ is*

$$\sigma_2(D^2) = - \sum_{i=1}^n \left(\frac{\partial}{\partial X^i} + \frac{1}{4} \sum_{j=1}^n R_{ij} X^j \right)^2 + \phi_2(\mathbf{F}) \in \mathfrak{P}(M, \mathcal{A}), \quad (4.6)$$

where $R_{ij} := \langle Re_i, e_j \rangle$ are the entries of the Riemann tensor and \mathbf{F} was defined in (4.2).

As our definition of the symbols is somewhat different to the one of both Roe and Getzler, we include a proof in appendix A.

For the other terms in the Witten operator $H_h = \hbar^2 D^2 + \hbar W + V$, it is clear that their q -symbols are given by

$$\phi_1(W) = \phi_{1,\bullet}(W) = \sum_{i=1}^n e^i \widehat{\otimes} \mathbf{b}(\nabla_i \xi) = \sum_{j=1}^n \partial_i \xi_j e^i \widehat{\otimes} \mathbf{b}(e^j) \quad (4.7)$$

$$\phi_0(V) = \phi_{0,\bullet}(V) = |\xi|^2. \quad (4.8)$$

We now need to extend the symbol calculus to the coefficients Φ_j of the formal heat kernel of H_h . We write short $\mathcal{E} = \Lambda T^* M$. For each fixed $q \in M$ and $t > 0$, $\Phi_j(\cdot, q, t)$ is a section of the vector bundle $\mathcal{E}^* \otimes \mathcal{E}_q$ over M . Therefore, with respect to a chart x around q and a trivialization that identifies fibers of \mathcal{E}_p with \mathcal{E}_q for p near q , it has a Taylor series

$$\Phi_j(\cdot, q, t) \sim \sum_{\alpha} \Phi_{j\alpha}(t) x^{\alpha}, \quad \text{where} \quad \Phi_{j\alpha}(t) \in \mathcal{E}_q^* \otimes \mathcal{E}_q \cong \text{End}(\mathcal{E}_q),$$

and the coefficients depend smoothly on t . Of course, we say that Φ_j has q -order k or less if $\Phi_{j\alpha}(t) \in \text{End}(\mathcal{E}_q)$ has Clifford order less or equal to $k + |\alpha|$ for each t , and in that case, we define its k -th symbol as

$$\sigma_k(\Phi_j) = \sum_{m+|\alpha|=k} \phi_m(\Phi_{j\alpha}) X^{\alpha} \in \mathcal{A}[T_p M].$$

The well-definedness can be shown as in the proof of 4.5 and a formal computation with Taylor series shows that we have the multiplication property

$$\sigma_k(P) \sigma_l(\Phi_j) = \sigma_{k+l}(P \Phi_j) \quad (4.9)$$

for $P \in \mathfrak{D}(M, \mathcal{E})$ of order at most k , which is supposed to act on the first entry of Φ_j .

Theorem 4.7. *For each $j = 0, 1, \dots$, Φ_j is of q -order at most j and for the "heat symbol" $\sigma(k_h) := \mathbf{e}_h \sum_{j=0}^n \hbar^j \sigma_j(\Phi_j)$, we have the formula*

$$\sigma(k_h) = u_h(X, R, t) \exp(-t \phi_0(V) - t \hbar \phi_1(W) - t \hbar^2 \phi_2(\mathbf{F})) \quad (4.10)$$

where

$$u_h(X, R, t) = (4\pi t\hbar^2)^{-n/2} \hat{A}(t\hbar R) \exp\left(-\frac{1}{4t\hbar^2} \left\langle X, \frac{t\hbar^2 R}{2} \coth\left(\frac{t\hbar^2 R}{2}\right) X \right\rangle\right)$$

is the Mehler kernel with the \hat{A} -form

$$\hat{A}(t\hbar R) = \det^{1/2} \left(\frac{t\hbar R/2}{\sinh t\hbar R/2} \right).$$

Let us see why this implies Thm. 4.1.

Proof (of 4.1). With a view on remark 4.4, we have

$$\sum_{j=0}^n \hbar^j \phi_{j,\bullet}(\Phi_j) = (4\pi t\hbar^2)^{n/2} \sigma(k_h)|_{X=0} = \hat{A}(t\hbar R) \exp(-t\phi_0(V) - t\hbar\phi_1(W) - t\hbar^2\phi_2(\mathbf{F})).$$

Furthermore $\hat{A}(t\hbar^2 R) = 1 + \sum_{j=1}^n \hbar^j \hat{A}_j(t)$ for some j -form $\hat{A}_j(t) \in \Omega^j(M)^2$ and the exp-part is of the form $\sum_{j=0}^n \hbar^j \sigma_{j,\bullet}(E_j(t))$ for some $E_j(t) \in \text{End}(\Lambda T_q^* M)$ of order (j, j) in the Clifford bi-filtration. Therefore

$$\sum_{j=0}^n \hbar^j \phi_{j,\bullet}(\Phi_j) = \sum_{j=0}^n \hbar^j \sum_{k+l=j} \hat{A}_k(t) \sigma_{l,\bullet}(E_l(t))$$

Because $\hat{A}_k(t)$ has order $(k, 0)$ in the Clifford bi-filtration, we now find that $\phi_{j,j}(\Phi_j) = \hat{A}_0(t) \phi_{j,j}(E_j(t)) = \phi_{j,j}(E_j(t))$. This is the proposition. \square

First we notice the following. For each $q \in M$, the "symbolic operator"

$$\sigma(H_h) := \hbar^2 \sigma_2(D^2) + \hbar \sigma_1(W) + \sigma_0(V) \quad (4.11)$$

is of Schrödinger type as an operator on $T_q M$, acting on C^∞ -functions with values in $\mathcal{A}_q = \Lambda T_q^* M \hat{\otimes} \text{End}(\Lambda T_q^* M)$ (i.e. sections of a trivial vector bundle). Therefore, we can investigate the heat equation

$$(\partial_t + \sigma(H_h)) \mathbf{k}_h = 0 \quad \text{on } T_q M. \quad (4.12)$$

We use bold letters for all objects associated to this symbolic equation to distinguish to the equation down on M . By 3.4, there exists a unique formal heat kernel

$$\hat{\mathbf{k}}_h(X, Y, t) = \mathbf{e}_h(X - Y, t) \sum_{j=0}^{\infty} \Phi_j(X, Y, t),$$

where the Φ_j solve the transport equations corresponding to the heat equation (4.12) and

$$\mathbf{e}_h(X, t) = (4\pi t\hbar^2)^{-n/2} \exp\left(-\frac{1}{4t\hbar^2} |X|^2\right).$$

²In fact, $\hat{A}_j(t) = 0$ unless j is divisible by four, as \hat{A} is a real characteristic class.

is the euclidean fundamental solution on $T_q M$.

Looking at formula (4.6), the unique connection ∇ on $T_q M$ such that $\sigma_2(D^2) - \nabla^* \nabla$ is of order zero clearly is given by

$$\nabla_i = \frac{\partial}{\partial X^i} + \sum_{j=1}^n R_{ij} X^j = \sigma_1(\nabla_i).$$

Because $T_q M$ is flat space, $\mathbf{G} \equiv 0$ and $\mathbf{V} = \sum_j X^j \partial/\partial X^j$, hence

$$\nabla_{\mathbf{V}} = \sum_{j=1}^n X^j \frac{\partial}{\partial X^j} + \sum_{ij=1}^n R_{ij} X^i X^j$$

which is just $\sigma_0(\nabla_{\mathbf{V}})$, as calculated in (A.2). The recursive transport equations (3.5) for the operator $\sigma(H)$ are therefore given by

$$\left(t \frac{\partial}{\partial t} + \sigma_0(\nabla_{\mathbf{V}}) + t \sigma_0(V) \right) \Phi_j(\cdot, Y) = -t \sigma_1(W) \Phi_{j-1}(\cdot, Y) - t \sigma_2(D^2) \Phi_{j-2}(\cdot, Y). \quad (4.13)$$

Lemma 4.8. *For each $j = 0, 1, \dots$, the coefficient Φ_j is of q -order at most j .*

Proof. By definition, for each $q \in M$, the Φ_j fulfill the transport equations

$$\left(t \frac{\partial}{\partial t} + \nabla_{\mathbf{V}} + G + tV \right) \Phi_j(\cdot, q) = -tW \Phi_{j-1}(\cdot, q) - tL \Phi_{j-2}(\cdot, q) \quad (4.14)$$

with initial condition $\Phi_0(q, q, 0) = \text{id}_{\mathcal{E}_q}$. Straightforward calculation shows that G is of order ≤ -1 , hence $\sigma_0(G) = 0$. Let Φ_0 be of q -order k . Taking the k -th symbol on both sides shows (by multiplicativity (4.9)) that $\sigma_k(\Phi_0)$ solves

$$\left(t \frac{\partial}{\partial t} + \sigma_0(\nabla_{\mathbf{V}}) + t \sigma_0(V) \right) \sigma_k(\Phi_0) = 0,$$

which is just the first transport equation on $T_q M$ (4.13). By Thm. 3.4, there is a unique solution for each initial value. Because $\Phi_0(q, q, 0) = \text{id}_{\mathcal{E}_q}$, we have

$$\sigma_k(\Phi_0)|_{(X,t)=0} = \phi_k(\Phi_0(q, q, 0)) = \phi_k(\text{id}_{\mathcal{E}_q}) = 0 \quad \text{if } k > 0,$$

so the initial value is zero and $\sigma_k(\Phi_0)$ vanishes for all X and t . This shows $k = 0$, i.e. Φ_0 has q -order zero for all t . Now by induction and multiplicativity of symbols, the right side of the j -th transport equation (4.14) has q -order $\leq j$. Suppose that Φ_j has q order $k > j$. Taking the k -th q -symbol on both sides shows that $\sigma_k(\Phi_j)$ solves

$$\left(t \frac{\partial}{\partial t} + \sigma_0(\nabla_{\mathbf{V}}) + t \sigma_0(V) \right) \sigma_k(\Phi_j) = 0.$$

Again, this is a transport equation. By 3.4, there is a unique solution for each right side, and zero is a solution, hence $\sigma_k(\Phi_j) = 0$ whenever $k > j$. But this means that Φ_j is of order $\leq j$. \square

Corollary 4.9. *The terms $\sigma_j(\Phi_j)$ solve the recursive transport equations (4.13) for $Y = 0$ and we have*

$$\Phi_j(\cdot, 0) = \sigma_j(\Phi_j). \quad (4.15)$$

Furthermore, because $\sigma_j(\Phi_j) = 0$ whenever $j > n$, the formal heat kernel $\hat{\mathbf{k}}_h(\cdot, 0, t)$ is actually a finite sum and is given by

$$\sigma(k_h) = \hat{\mathbf{k}}_h(\cdot, 0, t).$$

Proof. Taking the j -th q -symbol on both sides of equation j gives exactly the transport equations (4.13) for $\sigma(H_h)$. Equation (4.15) follows from the uniqueness statement of Thm. 3.4. \square

Let us now finish the proof of the theorem.

Proof (of Thm. 4.7). First, one verifies that the right side of (4.10) is a solution to the heat equation (4.12). The result that $u_h(X, R, t)$ satisfies

$$\left(\frac{\partial}{\partial t} - \sum_{i=1}^n \left(\frac{\partial}{\partial X^i} + \frac{1}{4} \sum_{j=1}^n R_{ij} X^j \right)^2 \right) u_t = 0, \quad u_0 = \delta_0 \cdot \text{id}_{\mathcal{E}_q}$$

is usually called Mehler's formula, see [BGV96, ch. 4.2]. Substitution $t \mapsto \hbar^2 t$ and straightforward calculation then shows that the right side of (4.10) solves (4.12).

Now explicitly expanding the Taylor series, one verifies that

$$u_h(X, R, t) \exp(-t\phi_0(V) - t\hbar\phi_1(W) - t\hbar^2\phi_2(\mathbf{F})) = \mathbf{e}_h(X, t)\Phi_h(X, t),$$

for some power series $\Phi_h = \sum_{j=0}^{\infty} \hbar^j \Phi_j$ with coefficients in $\mathcal{A}[T_q M]$, each Φ_j being a polynomial in both X and t . As seen in section 3, the Φ_j have to fulfill the recursive transport equations (4.13), and by uniqueness (theorem 3.4), we get

$$\Phi_h = \sum_{j=0}^n \hbar^j \sigma_j(\Phi_j)$$

as by 4.9, the $\sigma_j(\Phi_j)$ as well solve the transport equations. \square

5. The McKean-Singer Formula and its Consequences

From now on, let M be a compact Riemannian manifold (without boundary). Again, let D_h be Witten's perturbed Euler operator of example 3.2. The McKean-Singer formula [BGV96, Thm. 3.50] states that for all $t > 0$, we have

$$\text{ind}(D_h) = \int_M \text{str } k_h(t).$$

The index of the Euler operator $D = d + d^*$ on the exterior algebra equipped with the even-and-odd grading is well-known to be the Euler characteristic $\chi(M)$. On the other hand, the index is a topological invariant, i.e. is the same for every Dirac operator on a given vector bundle [BGV96, Thm. 3.51]. Therefore, the index of D_{\hbar} is also equal to $\chi(M)$. Expanding k_{\hbar} in its semiclassical expansion and using that the asymptotics are uniform over M by thm. 3.5, we get

$$\chi(M) \sim_{\hbar \searrow 0} (4\pi t \hbar^2)^{-n/2} \sum_{j=0}^{\infty} \hbar^j \int_M \text{str } \Phi_j(t)$$

Now the left side of this equation is independent of \hbar . By uniqueness of asymptotic expansions, all coefficients on the right side except the term constant in \hbar must vanish, and " \sim " must in fact be an equality. Whence we get

$$\chi(M) = (4\pi t)^{-n/2} \int_M \text{str } \Phi_n(t) \quad \text{for all } t > 0. \quad (5.1)$$

By prop. 2.3, the supertrace of Φ_n can be calculated in terms of the Clifford bi-symbol via the formula

$$\text{str}(\Phi_n) = (-1)^{n/2} 2^n \langle \phi_{n,n}(\Phi_n), \text{vol} \hat{\otimes} \text{vol} \rangle = (-1)^{n/2} 2^n \sum_{j=0}^n \langle \phi_{j,j}(\Phi_n), \text{vol} \hat{\otimes} \text{vol} \rangle \quad (5.2)$$

On the other hand, by theorem 4.1, we have

$$\sum_{j=0}^n \phi_{j,j}(\Phi_j) = \exp(-t|\xi|^2 - t\phi_{1,1}(W) - t\phi_{2,2}(\mathbf{F})).$$

where we used $\sigma_{0,0}(V) = |\xi|^2$. Therefore, we get the following theorem.

Theorem 5.1. *Let M be a compact Riemannian manifold of even dimension n . Then for all $t > 0$, we have*

$$\chi(M) = (4\pi t)^{-n/2} \int_M \text{str} \exp(-t|\xi|^2 - t\phi_{1,1}(W) - t\phi_{2,2}(\mathbf{F})). \quad (5.3)$$

Remark 5.2. The integrand above is in fact the (n -form coefficient of) the pullback of the Mathai and Quillen's Thom form [MQ86]

$$U = (2\pi)^{-n/2} T(\exp(-|\mathbf{x}|^2/2 + i\nabla\mathbf{x} + F)) \in \Omega^n(T^*M) \quad (5.4)$$

along the section $\xi_t := (t/2)^{1/2}\xi$. Regarding the terms appearing in (5.4), \mathbf{x} is the tautological section in $\Gamma^\infty(TM, \pi^*TM)$ that maps $\xi \mapsto \xi$, T is the Berezin integral on the second component. F is the Riemann tensor considered as $(2,2)$ -form and ∇ is the Levi-Civita connection on T^*M , both pulled back to T^*M via the canonical projection π (see [BGV96, ch. 1.6]).

Note that $\sigma_{1,1}(W)$ is not quite equal to $\nabla\xi$ because of the appearance of a super tensor product, whence the lack of the factor i in (5.3). By on possible definition, the Euler form is the pullback ι^*U along the inclusion of $\iota : M \longrightarrow T^*M$ as the zero section, which corresponds to $t = 0$. Therefore, (5.3) is exactly the interpolation formula

$$\chi(M) = \int_M \xi_t^* U,$$

which can be found in the proof of [BGV96, Thm. 1.56] (there however, one uses a vector field instead of a co-vector field).

In this sense, theorem 5.1 can be seen as a generalization of a result of Mathai [Mat92], who showed how to get the Thom form on the tangent bundle from the heat kernel of the Laplacian.

Let us now evaluate (5.3) without knowing anything about Thom forms. Note that $\phi_{2,2}(\mathbf{F})$, $\phi_{1,1}(W)$ and $|\xi|^2$ all commute so that

$$\exp(-t\phi_{2,2}(\mathbf{F}) - t\phi_{1,1}(W) - t|\xi|^2) = e^{-t|\xi|^2} \sum_{k=0}^{\infty} (-t)^k \sum_{j=0}^k \frac{\phi_{2,2}(\mathbf{F})^k \phi_{1,1}(W)^{k-j}}{k!(k-j)!}$$

Picking out the (n, n) -form part, we define the functions α_j by

$$\alpha_j \text{ vol } \widehat{\otimes} \text{ vol} := \frac{(-1)^j}{(n/2 - j)!(2j)!} \phi_{2,2}(\mathbf{F})^{n/2-j} \phi_{1,1}(W)^{2j} \quad (5.5)$$

so that

$$\phi_{n,n}(\Phi_n) = (-t)^{n/2} \sum_{j=0}^{n/2} t^j \alpha_j \text{ vol } \widehat{\otimes} \text{ vol}$$

Using formula (5.2), we get the following formula for the index density.

Lemma 5.3. *Let M be a compact Riemannian manifold of even dimension n . Then for all $t > 0$, we have*

$$\chi(M) = \pi^{-n/2} \sum_{j=0}^{n/2} t^j \int_M \alpha_j e^{-t|\xi|^2} \quad (5.6)$$

At this point, we make non-degeneracy assumptions on ξ and then evaluate this integral with the method of stationary phase. This allows us to express the Euler characteristic as a sum of integrals over the critical manifolds of ξ . The task is to explicitly calculate the functions α_j , at least at the points where ξ vanishes. This is quite easy for the extreme cases $j = 0$ and $j = n/2$ and a bit lengthy in the other cases. The latter is done in the next section.

Proposition 5.4. *The function α_0 is given by*

$$\alpha_0 = 2^{-n/2} \frac{(-1)^{n/2}}{2^n (n/2)!} \sum_{\tau, \sigma \in \mathcal{S}_n} \text{sgn}(\tau) \text{sgn}(\sigma) R_{\tau(1)\tau(2)\sigma(1)\sigma(2)} \cdots R_{\tau(n-1)\tau(n)\sigma(n-1)\sigma(n)}.$$

Proof. We have $\alpha_0 = 1/(n/2)! \phi_{2,2}(\mathbf{F})^{n/2}$, so the proposition follows directly from the formula $\sigma_{2,2}(\mathbf{F}) = -1/8 \sum_{ijkl} R_{ijkl} e^i e^j \widehat{\otimes} e^k e^l$ (compare (4.2)). \square

The term $K := 2^{n/2} \alpha_0$ is called Killing-Lipschitz curvature (or n -th order sectional curvature). Taking the limit $t \rightarrow 0$ in (5.6), we obtain the following classical theorem.

Corollary 5.5 (Gauss-Bonnet-Chern). [Che55, Thm. 1] *Let M be a compact Riemannian manifold of even dimension n . Then its Euler characteristic is given by the integral formula*

$$\chi(M) = (2\pi)^{-n/2} \int_M K, \quad (5.7)$$

where K is the Lipschitz-Killing curvature.

Proposition 5.6. *We have*

$$\alpha_{n/2} = \det_g(\nabla \xi)$$

where the determinant of the $(0, 2)$ tensor $\nabla \xi$ is calculated with help of the metric.

Proof. We have $\alpha_{n/2} \text{vol} \widehat{\otimes} \text{vol} = (-1)^{n/2}/n! w^n$. With respect to an orthonormal frame e^1, \dots, e^n , we have

$$w = \sum_{ij} \left(\partial_i \xi_j - \sum_k \Gamma_{ik}^j \xi_k \right) e^i \widehat{\otimes} e^j =: \sum_{ij} w_{ij} e^i \widehat{\otimes} e^j \quad (5.8)$$

Hence

$$(-1)^{n/2} w^n = \sum_{\tau, \sigma \in \mathcal{S}_n} \text{sgn}(\sigma) \text{sgn}(\tau) w_{\sigma(1)\tau(1)} \cdots w_{\sigma(n)\tau(n)} \text{vol} \widehat{\otimes} \text{vol} = n! \det(\nabla \xi) \text{vol} \widehat{\otimes} \text{vol},$$

where the factor $(-1)^{n/2}$ comes from the multiplication law of the super tensor product. \square

Taking the limit $t \rightarrow \infty$ in (5.6), we obtain the classical Poincaré-Hopf theorem.

Corollary 5.7 (Poincaré-Hopf). *Suppose that the vector field $X \in \Gamma^\infty(TM)$ is non-degenerate, i.e. whenever $X = 0$, then $\det(\nabla X) \neq 0$. Then we have*

$$\chi(M) = \sum_{\{X(p)=0\}} (-1)^{\nu(p)},$$

where $\nu(p)$ is the number of negative eigenvalues of ∇X at p .

This is a special case of Thm. 6.2. We give a separate proof, as it is quite short and hopefully gives an idea how the proof works in the general case.

Proof. Set $\xi := X^\flat$. Notice that the set C of points where $|X|^2 = |\xi|^2 = 0$ is just the set C_ϕ of critical points of $\phi := |X|^2$. The method of stationary phase (see e.g. [DS99, prop. 5.2] states that if ϕ fulfills $\phi > 0$ except at the point p , where $\phi(p) = 0$ and $\det_g \nabla^2 \phi > 0$, we have

$$\lim_{t \rightarrow \infty} t^{n/2} \int_M \alpha e^{-t\phi} = (2\pi)^{n/2} |\det \nabla^2 \phi|^{-1/2} \alpha(0)$$

for any compactly supported function α , while terms of lower order in t do not contribute. Here,

$$\nabla^2 \phi(X, Y) := \partial_X \partial_Y \phi - \partial_{\nabla_X Y} \phi \quad (5.9)$$

is the Hessian of ϕ . By a partition of unity argument, taking the limit $t \rightarrow \infty$ yields

$$\chi(M) = \lim_{t \rightarrow \infty} \pi^{-n/2} t^{n/2} \int_M \alpha_{n/2} e^{-t|X|^2} = 2^{n/2} \sum_{\{X(p)=0\}} (\det \nabla^2 |X|^2|_p)^{-1/2} \cdot \det(\nabla X_p)$$

Straightforward calculation shows that at p , $\nabla_{Y,Z}^2 |X|^2 = \langle \nabla_Y X, \nabla_Z X \rangle$ so that the determinant is given by $\det_g \nabla^2 |X|^2 = 2^n \det(\nabla X)^2$. Therefore

$$(\det \nabla^2 |X|^2)^{-1/2} \det(\nabla X) = 2^{-n/2} |\det(\nabla X)|^{-1} \det(\nabla X) = 2^{-n/2} \nu(p)$$

and the theorem follows. \square

6. The Degenerate Case

For simplicity, we assume that $\xi = d\phi$ for some function ϕ . From now on, we make the following assumptions on ϕ .

- i.) The set of critical points $C_\phi = \{p \in M \mid d\phi(p) = 0\}$ is a disjoint union of finitely many submanifolds of M .
- ii.) For each submanifold $C \subseteq C_\phi$ of M , the Hessian $\nabla^2 \phi$ of ϕ is non-degenerate when restricted to the normal bundle NC .

Definition 6.1. The index $\nu(C)$ of a connected critical submanifold $C \subseteq C_\phi$ of M is the dimension of the biggest subspace V of $T_p M$ such that the bilinear form $\nabla^2 \phi|_p$ restricted to V is negative definite.

This is independent of the point $p \in C$ because $\nu(C)$ is locally constant, as follows from a parameter-dependent version of the Morse lemma (see for example [Dui96, Lemma 1.2.2]).

This section is dedicated to proof the following theorem.

Theorem 6.2 (Degenerate Poincaré-Hopf). *Let M be an even-dimensional compact manifold. Suppose that ϕ fulfills the non-degeneracy assumptions above and let $C_\phi = C_1 \coprod \cdots \coprod C_k$ for connected submanifolds C_j . Then*

$$\chi(M) = \sum_{j=1}^k (-1)^{\nu(C_j)} \chi(C_j).$$

This theorem is also a direct consequence of the degenerate Morse inequalities [Bis86, Thm. 2.14].

Remark 6.3. Let $m_j = \dim(C_j)$. We will again use the method of stationary phase to show that there is an asymptotic expansion

$$\pi^{-n/2} \sum_{j=0}^{n/2} t^j \int_M a_j e^{-t|\xi|^2} \sim \sum_{j=0}^k \pi^{-m_j/2} \sum_{v=0}^{n/2} t^{v-(n-m_j)/2} \sum_{u=0}^{\infty} t^{-u} \int_{C_j} L_u \alpha_v$$

for some differential operators L_u (see lemma 6.5). By prop. 5.3, the left hand side is equal to $\chi(M)$ and hence independent of t . Therefore, all coefficients of the asymptotic series on the right must be zero except the terms constant in t . Working this out, we get

$$\chi(M) = \sum_{j=0}^k \pi^{-m_j/2} \int_{C_j} \sum_{s=0}^{m_j} L_{s/2} \alpha_{s/2+n/2-m_j/2}$$

The first observation is here that odd-dimensional manifolds do not contribute to the formula, which makes sense because also $\chi(C_j) = 0$ if m_j is odd.

The task is now to figure out the integrands $L_{s/2} \alpha_{s/2+n/2-m_j/2}$ on the critical submanifolds. By a partition of unity argument, we can and will assume from now on, that C_ϕ is already a submanifold of dimension $\dim(C_\phi) = m$ and that it is connected, so that $\nu(C_\phi)$ is well-defined. By 6.3, we may also assume that m is even.

We need to calculate the jets of α_j at C_ϕ for $2j \geq n - m$. To this end, we fix a point $p \in C_\phi$ and work in a geodesic chart x around p . In particular, this means that $\partial_1, \dots, \partial_n$ form an orthonormal basis at p . With respect to x , ϕ has the Taylor expansion

$$\phi \sim \frac{1}{2} \sum_{ij} \phi_{ij} x^i x^j + \frac{1}{6} \sum_{ijk} \phi_{ijk} x^i x^j x^k + \dots,$$

while the $(1,1)$ -form $\sigma_{1,1}(W)$ takes the form

$$\sigma_{1,1}(W) = \sum_{ij} \nabla^2 \phi(\partial_i, \partial_j) dx^i \hat{\otimes} dx^j = \sum_{ij} \underbrace{\left(\phi_{ij} + \sum_k \phi_{ijk} x^k + \dots \right)}_{:=w_{ij}} dx^i \hat{\otimes} dx^j$$

We will align x in such a way that the vectors $\partial_1, \dots, \partial_m$ are tangent to C_ϕ at p , the vectors $\partial_{m+1}, \dots, \partial_n$ are normal to C_ϕ at p and that furthermore the Hessian of ϕ takes

the form

$$\nabla^2 \phi \hat{=} (\phi_{ij})_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix} \quad \text{with} \quad \Lambda = \begin{pmatrix} \lambda_{m+1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}. \quad (6.1)$$

For abbreviation, we write $e_j := \partial_j|_p$. We will come back to this situation throughout the whole section.

Lemma 6.4. *Let $0 \leq s \leq m$ be even and $j := s/2 + n/2 - m/2$. For each $p \in C_\phi$, the function α_j defined in (5.5) is of p -order at most $-s$ (i.e. it vanishes to order s at p) and its p -symbol is given by*

$$\sigma_{-s}(\alpha_j) = \frac{(-8)^{s/2-m/2}}{s!(m/2-s/2)!} \det(\nabla^2 \phi|_{NC_\phi}) \theta^s(X),$$

where $\nabla^2 \phi|_{NC_\phi}$ is the Hessian of ϕ restricted to the normal bundle and $\theta^s(X)$ is the homogeneous polynomial $\theta^s(X) = \sum_{u_1, \dots, u_s=1}^n \theta_{u_1 \dots u_s}^s X^{u_1} \dots X^{u_s} \in \mathbb{R}[T_p M]$ with the coefficients

$$\theta_{u_1 \dots u_s}^s = \sum_{\tau, \sigma \in S_m} \text{sgn}(\tau) \text{sgn}(\sigma) \prod_{k=1}^s \phi_{\tau(k)\sigma(k)} u_k \prod_{k=s/2+1}^{m/2} R_{\sigma(2k-1)\sigma(2k)\tau(2k-1)\tau(2k)}$$

Proof. Remember that

$$\alpha_j = \frac{(-1)^j}{(n/2-j)!(2j)!} \left\langle \phi_{2,2}(\mathbf{F})^{n/2-j} \phi_{1,1}(W)^{2j}, \text{vol} \hat{\otimes} \text{vol} \right\rangle.$$

We have

$$\phi_{1,1}(W)^{2j} = (-1)^j \sum_{|I|, |J|=2j} w_{IJ} e^I \hat{\otimes} e^J, \quad \text{where} \quad w_{IJ} := w_{i_1 j_1} \dots w_{i_{2j} j_{2j}}$$

and I and J range over all $2j$ -tuples of numbers between 1 and n . As seen in (6.1), $w_{ij}(p) = 0$ whenever $i \leq m$ or $j \leq m$, so the term w_{IJ} has p -order $-r$ if $i_k \leq m$ or $j_k \leq m$ for at least r numbers k . Conversely, w_{IJ} can have p -order at most $-s = n - m - 2j$ and this is the case when $i_k > m$ and $j_k > m$ for exactly $n - m$ numbers k between 1 and $2j$. Therefore,

$$(-1)^j \sum_{|I|, |J|=2j} \sigma_{-s}(w_{IJ}) e^I \hat{\otimes} e^J = (-1)^j \binom{2j}{n-m} \sum_{I, J \in \mathcal{J}} \sigma_{-s}(w_{IJ}) e^I \hat{\otimes} e^J,$$

where \mathcal{J} is the set of all tuples $I = (i_1, \dots, i_{2j})$ of numbers between 1 and n such that $i_1, \dots, i_s \leq m$ and $i_{s+1}, \dots, i_{2j} > m$. For $k, l > m$, we have $w_{kl} = \phi_{kl}$. Denoting by \mathcal{J}_2 the set of tuples $I = (i_{s+1}, \dots, i_{2j})$ of $n - m$ numbers $m < i_k \leq n$ and by $\mathcal{S}_n/\mathcal{S}_m$ the group of

permutations of the numbers $(m+1, \dots, n)$, we have

$$\begin{aligned}
\sum_{I, J \in \mathcal{J}_2} w_{IJ} e^I \widehat{\otimes} e^J &= \sum_{\sigma, \tau \in \mathcal{S}_n / \mathcal{S}_m} \text{sgn}(\tau) \text{sgn}(\sigma) \prod_{k=m+1}^n \phi_{\sigma(k)\tau(k)} e^{m+1} \dots e^n \widehat{\otimes} e^{m+1} \dots e^n \\
&= (n-m)! \det_{m < k, l \leq n} (\phi_{kl}) e^{m+1} \dots e^n \widehat{\otimes} e^{m+1} \dots e^n \\
&= (n-m)! \det(\nabla^2 \phi|_{NC_\phi}) e^{m+1} \dots e^n \widehat{\otimes} e^{m+1} \dots e^n.
\end{aligned}$$

Therefore

$$\begin{aligned}
(-1)^j \sum_{|I|, |J|=2j} \sigma_{-s}(w_{IJ}) e^I \widehat{\otimes} e^J &= (-1)^j \binom{2j}{n-m} \sum_{I_1, J_1 \in \mathcal{J}_1} \sum_{I_2, J_2 \in \mathcal{J}_2} \sigma_{-s}(w_{I_1 J_2}) w_{I_2 J_2} e^{I_1} e^{I_2} \widehat{\otimes} e^{J_1} e^{J_2} \\
&= (-1)^j \underbrace{\frac{(2j)!}{(2j-n+m)!}}_s \det(\nabla^2 \phi|_{NC_\phi}) \sum_{I, J \in \mathcal{J}_1} \sigma_{-s}(w_{IJ}) e^I e^{m+1} \dots e^n \widehat{\otimes} e^J e^{m+1} \dots e^n.
\end{aligned}$$

where \mathcal{J}_1 is the set of tuples $I = (i_1, \dots, i_s)$ of numbers $1 \leq i_k \leq m$. The matrix in the determinant is just the Hessian of ϕ restricted to the normal directions. Multiplying with $\phi_{2,2}(\mathbf{F})^{n/2-j}$ then gives

$$\begin{aligned}
\sigma_{-s}(\phi_{2,2}(\mathbf{F})^{n/2-j} \phi_{1,1}(W)^{2j}) &= \phi_{2,2}(\mathbf{F})^{n/2-j} \sigma_{-s}(\phi_{1,1}(W)^{2j}) \\
&= (-1)^j \frac{(2j)!}{s!} \left(-\frac{1}{8}\right)^{m/2-s/2} \det(\nabla^2 \phi|_{NC_\phi}) \theta^s \text{vol} \widehat{\otimes} \text{vol}
\end{aligned}$$

where

$$\theta^s = \sum_{\tau, \sigma \in \mathcal{S}_n} \text{sgn}(\tau) \text{sgn}(\sigma) \prod_{k=1}^s \sigma_{-1}(w_{\tau(k)\sigma(k)}) \prod_{k=s/2+1}^{m/2} R_{\tau(2k-1)\tau(2k)\sigma(2k-1)\sigma(2k)}.$$

Now remember that for $k, l \leq m$, we have $\sigma_{-1}(w_{kl}) = \sum_{u=1}^n \phi_{klu} X^u$. The proposition follows after multiplying with $(-1)^j / (2j)!(n/2-j)!$. \square

Lemma 6.5. *Let $0 \leq s \leq m$ be even and $j := s/2 + (n-m)/2$. Let $\alpha \in C^\infty(M)$ be of p -order $-s$ at each $p \in C_\phi$ (i.e. α vanishes order s on C_ϕ). Then*

$$\lim_{t \rightarrow \infty} \pi^{-n/2} t^j \int_M \alpha e^{-t|\text{d}\phi|^2} = \frac{\pi^{-m/2}}{2^s (s/2)!} \int_{C_\phi} \frac{1}{|\det \nabla^2 \phi|_{NC_\phi}} L_\phi^{s/2} \alpha, \quad (6.2)$$

for each differential operator L_ϕ of degree 2 on M such that with respect to the coordinates above, the second p -symbol of L_ϕ in the coordinates above is given by

$$\sigma_2(L_\phi) = \sum_{k=m+1}^n \frac{1}{\lambda_k^2} \frac{\partial^2}{\partial X_k^2}.$$

Proof. First notice that if the proposition holds for one such operator L_ϕ , then it holds for all, because as α vanishes to order s on C_ϕ while $L_\phi^{s/2}$ is of order s , only the highest order term contributes to the result.

Choose a submanifold chart $y = (y_1, y_2)$ for C_ϕ , defined on a neighborhood U of p , such that the vectors $\partial_{y^1}, \dots, \partial_{y^m}$ are tangent to C_ϕ and the vectors $\partial_{y^{m+1}}, \dots, \partial_{y^n}$ are normal to C_ϕ for all $q \in C_\phi \cap U$. With respect to this chart y , the Hessian in normal direction $\nabla^2|\mathrm{d}\phi|^2|_{NC_\phi}$ is on $C_\phi \cap U$ given by the matrix

$$Q(y_1) = (Q_{ij}(y_1))_{ij} = \left(\frac{\partial^2 |\mathrm{d}\phi|^2}{\partial y^i \partial y^j} \right)_{ij}.$$

In fact, we can even arrange y in such a way that

$$|\mathrm{d}\phi|^2 = \frac{1}{2} \langle Q(y_1) y_2, y_2 \rangle.$$

For a proof of this statement, see e.g. [Dui96, lemma 1.2.2.].

It is standard that the integral (6.2) is of order $t^{-\infty}$ if α vanishes on a neighborhood of C_ϕ . Therefore, with a partition of unity argument, we may assume that α has support in the domain of the chart y . Now

$$\int_M \alpha e^{-t|\mathrm{d}\phi|^2} = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n-m}} \alpha(y) G(y) e^{-t \frac{1}{2} \langle Q(y_1) y_2, y_2 \rangle} dy_2 dy_1, \quad (6.3)$$

where $G(y) = \det^{1/2}(g_{ij}(y))$. For $t \rightarrow \infty$, we use the method of stationary phase in \mathbb{R}^n [Dui96, Prop. 1.2.4.] to get the asymptotic expansion

$$\int_{\mathbb{R}^{n-m}} \alpha(y) G(y) e^{-t \frac{1}{2} \langle Q(y_1) y_2, y_2 \rangle} dy_2 \sim \frac{(2\pi/t)^{(n-m)/2}}{|\det Q(y_1)|^{1/2}} \sum_{k=0}^{\infty} \frac{t^{-k}}{k!} L^k[\alpha \cdot G](y_1, 0), \quad (6.4)$$

depending smoothly on y_1 , where L is the differential operator

$$L(y_1) = \frac{1}{2} \sum_{ij=m+1}^n Q^{ij}(y_1) \frac{\partial^2}{\partial y^i \partial y^j}$$

featuring the inverse matrix (Q^{ij}) of (Q_{ij}) . Because α is of q -order $-s$, the first non-vanishing term in (6.4) is the one with $k = -(n-m)/2 - s/2 = -j$. Therefore,

$$\lim_{t \rightarrow \infty} t^j \int_{\mathbb{R}^{n-m}} \alpha(y) G(y) e^{-t \frac{1}{2} \langle Q(y_1) y_2, y_2 \rangle} dy_2 = \frac{(2\pi)^{(n-m)/2}}{(s/2)! |Q(y_1)|^{1/2}} L^{s/2}[\alpha \cdot G](y_1, 0).$$

and

$$\int_M \alpha e^{-t|\mathrm{d}\phi|^2} = \int_{\mathbb{R}^m} \frac{(2\pi)^{(n-m)/2}}{(s/2)! |Q(y_1)|^{1/2}} L^{s/2}[\alpha \cdot G](y_1, 0) dy_1$$

Let us verify that this integrand coincides pointwise with that of the proposition. First, straightforward calculation shows that on C_ϕ , $\nabla_{X,Y}^2 |\mathrm{d}\phi|^2 = 2 \langle \nabla^2 \phi(X, \cdot), \nabla^2 \phi(Y, \cdot) \rangle$ so that

$$\det Q(y_1) = \det(\nabla^2 |\mathrm{d}\phi|^2|_{NC_\phi}) = 2^{(n-m)/2} (\det \nabla^2 \phi|_{NC_\phi})^2.$$

Now fix $p \in C_\phi$ and arrange the submanifold chart y in such a way that it coincides to first order with the geodesic chart x from above, i.e. $y(p) = 0$ and $\partial_{y^j}|_p = \partial_{x^j}|_p$ for $j = 1, \dots, n$ (this can be done by composition with a suitable affine transformation of \mathbb{R}^n). Then $Q_{ij}(0) = 0$ for $i \neq j$ and $Q_{jj}(0) = 2\lambda_{j+m}^2$, $Q^{jj}(0) = 1/2\lambda_{j+m}^2$ for $j = 1, \dots, n-m$. Therefore, the p -symbol is given by

$$\sigma_s(L^{s/2}) = \left(\frac{1}{2} \sum_{ij=1}^{n-m} Q^{ij} \frac{\partial^2}{\partial X^i \partial X^j} \right)^{s/2} = 2^{-s/2} \left(\sum_{j=m+1}^n \frac{1}{2\lambda_j^2} \frac{\partial^2}{\partial X_j^2} \right)^{s/2} = 2^{-s} \sigma_2(L_\phi)^{s/2}.$$

Furthermore, $G(p) = 1$ as y coincides with a geodesic chart up to first order, so at p ,

$$\sigma_0(L^{s/2}[\alpha \cdot G]) = \sigma_s(M_{s/2})\sigma_{-s}(\alpha)\sigma_0(G) = \sigma_s(L_\phi^{s/2})\sigma_{-s}(\alpha) = \sigma_0(L_\phi^{s/2}\alpha)$$

which shows that $L^{s/2}[\alpha \cdot G] = L_\phi^{s/2}\alpha$ at p . \square

Lemma 6.6. *In terms of the chart x above, the second fundamental form of C_ϕ at the point p is given by*

$$\Pi(\partial_i, \partial_j) = - \sum_{k=m+1}^n \frac{1}{\lambda_k} \phi_{ijk} \partial_k, \quad i, j \leq m.$$

Proof. Near p , set

$$f_i := \partial_i \phi = \lambda_i x^i + \frac{1}{2} \sum_{jk} \phi_{ijk} x^j x^k + \dots, \quad i = m+1, \dots, n$$

Then there exists an open neighborhood U of p in M such that

$$C_\phi \cap U = \{p \in U \mid f_{m+1}(p) = \dots = f_n(p) = 0\}.$$

Then the vector fields $\nu_i := \text{grad } f_i$ are normal to C_ϕ . Choose a local frame ν_1, \dots, ν_m of TC_ϕ such that $\nu_j|_p = \partial_j|_p$ and extend it to a neighborhood of $C_\phi \cap U$ in U (for example via parallel translation along ν_j , $j > m$). Now on $C_\phi \cap U$, we have $\langle \nu_j, \nu_k \rangle = 0$ whenever $j \leq m$ and $k > m$. So $\langle \nu_j, \nu_k \rangle$ is constant in tangent directions, which means that for $i \leq m$,

$$0 = \partial_i \langle \nu_j, \nu_k \rangle = \langle \nabla_i \nu_j, \nu_k \rangle + \langle \nu_j, \nabla_i \nu_k \rangle.$$

From this follows

$$\langle \Pi(\nu_i, \nu_j), \nu_k \rangle = \langle \nabla_i \nu_j, \nu_k \rangle = -\langle \nu_j, \nabla_i \nu_k \rangle = -\nabla^2 f_k(\nu_i, \nu_j) \stackrel{\text{in } p}{=} -\phi_{kij}$$

Now at p , we have $\nu_i = \partial_i$ and $\nu_j = \partial_j$ for $i, j \leq m$ as well as $\nu_k = \lambda_k \partial_k$ for $k > m$, so that at p

$$\langle \Pi(\partial_i, \partial_j), \partial_k \rangle = -\frac{1}{\lambda_k} \nabla^2 f_k(\partial_i, \partial_j) = -\frac{1}{\lambda_k} \phi_{kij} = -\frac{1}{\lambda_k} \phi_{ijk} \quad \square$$

Remark 6.7. The Gauss formula [Cha06, Thm. II.2.1] states that if \tilde{R}_{ijkl} are the entries of the Riemann tensor of C_ϕ with respect to the basis $\partial_1, \dots, \partial_m$, then they are given by the formula

$$\tilde{R}_{ijkl} = R_{ijkl} - S_{ijkl},$$

where R_{ijkl} is the Riemann tensor of M and

$$S_{ijkl} = \langle \Pi(\partial_i, \partial_k), \Pi(\partial_j, \partial_l) \rangle - \langle \Pi(\partial_j, \partial_k), \Pi(\partial_i, \partial_l) \rangle.$$

By 6.6, the tensor S at p has the entries

$$S_{ijkl} = \sum_{u=1}^n \frac{1}{\lambda_u^2} (\phi_{iku} \phi_{jlu} - \phi_{jku} \phi_{ilu}). \quad (6.5)$$

Lemma 6.8. Let $0 \leq s \leq m$ be even, $j := s/2 + n/2 - m/2$ and let L_ϕ be an operator as in 6.5. Then

$$L_\phi^{s/2} \alpha_j = \frac{(-8)^{s/2-m/2} 2^{-s/2}}{(m/2 - s/2)!} \det(\nabla^2 \phi|_{NC_\phi}) \Theta^s$$

where

$$\Theta^s = \sum_{\tau, \sigma \in \mathcal{S}_m} \text{sgn}(\tau) \text{sgn}(\sigma) \prod_{k=1}^{s/2} S_{\tau(2k+1)\tau(2k)\sigma(2k+1)\sigma(2k)} \prod_{k=s/2+1}^{m/2} R_{\tau(2k+1)\tau(2k)\sigma(2k+1)\sigma(2k)}.$$

Proof. We need to calculate $\sigma_s(L_\phi^{s/2})\theta^s(X)$, where $\theta^s(X) = \sum \theta_{u_1 \dots u_s}^s X^{u_1} \dots X^{u_s}$ is the homogeneous polynomial of degree s of lemma 6.4. There are $2^{-s/2}s!$ possibilities to divide s indices into pairs of two. Therefore

$$\begin{aligned} \sigma_s(L_\phi^{s/2})\theta^s(X) &= 2^{-s/2}s! \sum_{u_1, \dots, u_s=1}^n \theta_{u_1 \dots u_s}^s \sigma_2(L_\phi)[X^{u_1} X^{u_2}] \dots \sigma_2(L_\phi)[X^{u_{s-1}} X^{u_s}] \\ &= s! \sum_{k_1, \dots, k_{s/2}=m+1}^n \frac{1}{\lambda_{k_1}^2} \dots \frac{1}{\lambda_{k_{s/2}}^2} \theta_{k_1 k_1 \dots k_{s/2} k_{s/2}}^s \end{aligned}$$

When written out this equals $s!$ times

$$\sum_{\tau, \sigma \in \mathcal{S}_m} \text{sgn}(\tau) \text{sgn}(\sigma) \prod_{k=1}^{s/2} \sum_{u=m+1}^n \frac{1}{\lambda_u^2} \phi_{\tau(2k-1)\sigma(2k-1)u} \phi_{\tau(2k)\sigma(2k)u} \prod_{k=s/2+1}^{m/2} R_{\sigma(2k-1)\sigma(2k)\tau(2k-1)\tau(2k)}.$$

Anti-symmetrization shows with a view on (6.5) that this is equal to $2^{-s/2}\Theta^s$ so the proposition follows from the formula in lemma 6.4. \square

The proof of theorem 6.2 is now almost complete.

Proof (of 6.2). If we write $j = s/2 + n/2 - m/2$, we have by (5.6) and lemma 6.5

$$\begin{aligned}
\chi(M) &= \sum_{s=0}^{m/2} \frac{\pi^{-m/2}}{2^s(s/2)!} \int_{C_\phi} \frac{1}{|\det \nabla^2 \phi|_{NC_\phi}|} L_\phi^{s/2} \alpha_j \\
&= \sum_{s=0}^{m/2} \frac{\pi^{-m/2}}{2^s(s/2)!} \frac{(-8)^{s/2-m/2} 2^{-s/2}}{(m/2-s/2)!} \int_{C_\phi} \frac{\det \nabla^2 \phi|_{NC_\phi}}{|\det \nabla^2 \phi|_{NC_\phi}|} \Theta^s \\
&= (-1)^{\nu(C_\phi)} (2\pi)^{-m/2} \frac{(-1)^{m/2}}{2^m(m/2)!} \int_{C_\phi} \sum_{s=0}^{m/2} (-1)^{s/2} \binom{m/2}{s/2} \Theta^s \\
&= (-1)^{\nu(C_\phi)} (2\pi)^{-m/2} \int_{C_\phi} \tilde{K},
\end{aligned}$$

where \tilde{K} is the Lipschitz-Killing curvature of C_ϕ . To see the last step, we observe in view of remark 6.7 that

$$\tilde{R}^{n/2} = \sum_{s=0}^{m/2} (-1)^{s/2} \binom{m/2}{s/2} S^{s/2} R^{m/2-s/2} = \sum_{s=0}^{m/2} (-1)^{s/2} \binom{m/2}{s/2} \Theta^s \text{vol} \hat{\otimes} \text{vol}$$

Here, we wrote

$$\begin{aligned}
S &= \sum_{ijkl} S_{ijkl} e^i e^k \hat{\otimes} e^j e^l \quad \text{and} \\
R &= -8 \sigma_{2,2}(\mathbf{F}) = \sum_{ijkl} R_{ijkl} e^i e^k \hat{\otimes} e^j e^l.
\end{aligned}$$

Now theorem 6.2 follows with the Gauss-Bonnet-Chern theorem 5.5. \square

Remark 6.9. Let us conclude this section of lengthy calculations with some "philosophic" remarks. The method of stationary phase asserts that in the case that the critical set of ϕ is a non-degenerate submanifold C of dimension m , there is an asymptotic expansion of numbers

$$\int_M \alpha e^{-t|\mathrm{d}\phi|^2} \sim (2\pi/t)^{(n-m)/2} \sum_{j=0}^{\infty} t^{-j} a_j$$

as $t \rightarrow \infty$ and also provides a method how to represent the numbers a_j as integrals

$$a_j = \int_C \mu_j,$$

where μ_j is some density on C . However, although the numbers a_j are uniquely determined by M , α and ϕ , the densities μ_j are not. Even in the case that M is Riemannian, one has too many degrees of freedom in the choice of the submanifold chart y that appeared in the proof of lemma 6.5. Taking another such chart, one ends up with a different operator L_ϕ (though all these operators coincide to top order) and another metric function G , all

this altering the density μ_j (however, the integral of μ_j over C must be equal to a_j for each choice of y).

The interesting part in the calculation above is the observation that despite all arbitrariness, we still get canonical candidates for the relevant μ_j 's. This comes from the fact that the integrand α vanishes exactly to order s when applied with $L_\phi^{s/2}$, thus making only the top order terms of α and $L_\phi^{s/2}$ relevant (which are canonically given by the Riemannian structure). To give a sketchy summary, there is a "magic cancellation of arbitrariness" under way when it comes to the relevant terms.

A. The Symbol of the Dirac Operator

For convenience, we show how to derive the Getzler symbol of the Dirac operator proposed in theorem 4.6. Let us start by recalling that the Weizenböck formula for the Euler operator $D = d + d^*$ is [BGV96, Thm. 3.52]

$$\Delta = D^2 = \nabla^* \nabla + \frac{\text{scal}}{4} + \mathbf{F}, \quad (\text{A.1})$$

where

$$\mathbf{F} = -\frac{1}{8} \sum_{ijkl} R_{ijkl} \mathbf{c}^i \mathbf{c}^j \mathbf{b}^k \mathbf{b}^l \in \text{End}(\Lambda T^* M)$$

Lemma A.1. *Let ∇ be the Levi-Civita connection on $\Lambda T^* M$ and let e_1, \dots, e_n is a local orthonormal basis near $q \in M$. Then the operator ∇_i is of q -order 1 and*

$$\sigma_1(\nabla_i) = \frac{\partial}{\partial X^i} + \sum_{j=1}^n R_{ij} X^j \in \mathfrak{P}(T_q M, \Lambda T_q^* M), \quad (\text{A.2})$$

where $R_{ij} = \langle R e_i, e_j \rangle = \sum_{k < l} R_{ijkl} e^k \wedge e^l$ are the entries of the curvature tensor of M with respect to this basis.

Proof. With respect to the chosen orthonormal basis, we have $\nabla_i = \partial/\partial x^i + \Gamma_i$, where Γ_i are the Christoffel symbols on $\Lambda T^* M$. By [LM89, (4.34)],

$$\Gamma_i = -\frac{1}{2} \sum_{l < k} \Gamma_{ik}^l (\mathbf{c}^k \mathbf{c}^l - \mathbf{b}^k \mathbf{b}^l), \quad (\text{A.3})$$

where Γ_{ik}^l are the Christoffel symbols of the Levi Civita connection on TM . As we make only pointwise computations, we may assume our framing e_1, \dots, e_n to be synchronous in q ; then we have using orthogonality of the framing [BGV96, (1.23)]

$$\Gamma_{ik}^l = -\frac{1}{2} \sum_{j=1}^n R_{kij}^l x^j + \text{lower order} = -\frac{1}{2} \sum_{j=1}^n R_{klij} x^j + \text{lower order}. \quad (\text{A.4})$$

In total, we get that Γ_i is in fact of order 1 and

$$\sigma_1(\nabla_i) = \frac{\partial}{\partial X^i} + \sigma_1(\Gamma_i) = \frac{\partial}{\partial X^i} + \frac{1}{4} \sum_{j=1}^n \sum_{k < l} R_{klij} e^k \wedge e^l X^j = \frac{\partial}{\partial X^i} + \frac{1}{4} \sum_{j=1}^n R_{ij} X^j$$

using $R_{klij} = R_{ijkl}$. \square

Corollary A.2. *For each $q \in M$, the operator $\nabla_{\mathcal{V}}$ appearing in the transport equations (3.5) is of order 0 and its q -symbol is*

$$\sigma_0(\nabla_{\mathcal{V}}) = \sum_{j=1}^n X^j \frac{\partial}{\partial X^j} + \sum_{ij} R_{ij} X^i X^j. \quad (\text{A.5})$$

Proof. If x is a geodesic chart around $q \in M$, we have $d(\cdot, q)^2 = |x|^2$ near q , so that $\mathcal{V} = \frac{1}{2} \text{grad}(|x|^2)$. Straightforward calculation shows

$$\mathcal{V} = \sum_{i=1}^n x^i \partial_i + \text{lower order terms.}$$

Hence for a vector field Y , we have

$$\nabla_{\mathcal{V}} Y = \sum_{i,k=1}^n x^i \left(\frac{\partial Y^k}{\partial x^i} + \sum_{j=1}^n \Gamma_{ij}^k Y^j \right) \partial_k + \text{lower order terms.}$$

and the proposition follows from (A.4). \square

Now we can proof Theorem 4.6.

Proof (of Thm. 4.6). We have the formula $\nabla^* \nabla = -g^{ij} (\nabla_i \nabla_j - \Gamma_{ij}^k \nabla_k)$ with respect to a local frame, where Γ_{ij}^k are the Christoffel symbols of the Levi Civita connection on TM [BGV96, p.66]. If we choose an orthogonal framing that is synchronous at q , the Christoffel symbols vanish at q and therefore $\sigma_2(\nabla^* \nabla) = -\sum_i \sigma_1(\nabla_i)^2$.

Using the Weizenböck formula (A.1), we get

$$\sigma_2(D^2) = -\sum_{i=1}^n \sigma_1(\nabla_i^{\mathcal{E}})^2 + \frac{1}{4} \sigma_2(\text{scal}) + \sigma_2(\mathbf{F}).$$

scal has q -order zero so its second symbol vanishes, while $\sigma_2(\mathbf{F}) = \phi_2(\mathbf{F})$. The formula for $\sigma_2(D^2)$ is now a consequence of A.2. \square

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